Lecture Notes

On

Transportation and Assignment Problems

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Quantitative Techniques for Managers

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The transportation problem is a special type of linear programming problem where the objective is to minimise the cost of distributing a product from a number of **sources** or **origins** to a number of **destinations.** Because of its special structure the usual simplex or graphical method is not suitable for solving transportation problems. These problems require a special method of solution. The **origin** of a transportation problem is the location from which shipments are despatched. The **destination** of a transportation problem is the location to which shipments are transported. The **unit transportation cost** is the cost of transporting one unit of the consignment from an origin to a destination.

The transportation problem in operational research is concerned with finding the minimum cost of transporting a single commodity from a given number of sources (e.g. factories) to a given number of destinations (e.g. warehouses). These types of problems can be solved by general network methods, but here we use a specific transportation algorithm.

The data of the model include

1. The level of supply at each source and the amount of demand at each destination.

2. The unit transportation cost of the commodity from each source to each destination.

Since there is only one commodity, a destination can receive its demand from more than one source. The objective is to determine how much should be shipped from each source to each destination so as to minimise the total transportation cost.

Types of Transportation Problem

1. Balanced Transportation Problem
2. Unbalanced Transportation Problem

Solution of the transportation problem

Stage I: Finding an initial basic feasible solution.

Stage II: Checking for optimality

Existence of Feasible Solution: A necessary and sufficient condition for the existence of a feasible solution to the general transportation problem is that

Total supply = Total demand

Existence of Basic Feasible Solution: The number of basic variables of the general transportation problem at any stage of feasible solution must be $(m + n - 1)$. Now degenerate basic feasible solution (a feasible solution) involving exactly $(m + n - 1)$ positive variables is known as non-degenerate basic feasible solution otherwise it is said to be degenerate basic feasible. These allocations should be independent positions in case of non-degenerate basic feasible solutions.

There are 3 methods of finding initial basic feasible solution-

1. Northwest corner method
2. Least cost method
3. Vogel's approximation method (or Penalty method)

Steps for North-West Corner Method

1. Allocate the maximum amount allowable by the supply and demand constraints to the variable x_{11} (i.e. the cell in the top left corner of the transportation tableau).
2. If a column (or row) is satisfied, cross it out. The remaining decision variables in that column (or row) are non-basic and are set equal to zero. If a row and column are satisfied simultaneously, cross only one out (it does not matter which).
3. Adjust supply and demand for the non-crossed out rows and columns.
4. Allocate the maximum feasible amount to the first available non-crossed out element in the next column (or row).
5. When exactly one row or column is left, all the remaining variables are basic and are assigned the only feasible allocation.

Steps for Least Cost Method

1. Assign as much as possible to the cell with the smallest unit cost in the entire tableau. If there is a tie then choose arbitrarily.
2. Cross out the row or column which has satisfied supply or demand. If a row and column are both satisfied then cross out only one of them.
3. Adjust the supply and demand for those rows and columns which are not crossed out.
4. When exactly one row or column is left, all the remaining variables are basic and are assigned the only feasible allocation.

Steps for Vogel's Approximation Method

1. Determine a ***penalty cost*** for each row (column) by subtracting the lowest unit cell cost in the row (column) from the *next lowest unit cell cost* in the same row (column).
2. Identify the row or column with the greatest penalty cost. Break the ties arbitrarily (if there are any). Allocate as much as possible to the variable with the lowest unit cost in the selected row or column. Adjust the supply and demand and cross out the row or column that is already satisfied. If a row and column are satisfied simultaneously, only cross out one of the two and allocate a supply or demand of zero to the one that remains.

- If there is exactly one row or column left with a supply or demand of zero, stop.
- If there is one row (column) left with a *positive* supply (demand), determine the basic variables in the row (column) using the Minimum Cell Cost Method. Stop.
- If all of the rows and columns that were not crossed out have zero supply and demand (remaining), determine the basic *zero* variables using the Minimum Cell Cost Method. Stop.
- In any other case, continue with Step 1.

Unbalanced Transportation Model

A necessary and sufficient condition for the existence of feasible solution to the general transportation problem is that the total demand must equal the total supply. However, sometimes there may be more demand than the supply and vice versa in which case the problem is said to be unbalanced. If may occur in the following situation

1. SS > DD

2. DD > SS

In case I, we introduce a dummy destination in the transportation table. The cost of transportating to this destination are all set equal to zero. The requirement at this dummy destination is then assumed to be equal to SS – DD

In case 2, we introduce a dummy source in the transportation table. The cost of transporting from this source to any destinations is all set equal to zero. The availability at this dummy source is assumed to be equal to DD – SS.

Test of optimality and optimal solution

Once an initial solution is obtained, the next step is to check its optimality. An optimal solution is one where there is no other set of transportation route (allocations) that will further reduce the total transportation cost. Thus we have to evaluate each unoccupied cell (represents unused route) in the transportation table in terms of an opportunity of reducing total transportation cost. There are two methods to check optimality

1. Stepping Stone Method

2. Modified distribution (MODI)

Stepping Stone Method

This is a procedure for determining the potential of improving upon each of the non-basic variables in terms of the objective function. To determine this potential, each of the non-basic variables is considered one by one. For each such cell, we find what effect on the total cost would be if one unit is assigned to this cell. With this information, then, we come to know whether the solution is optimal or not. If not, we improve that solution.

We can summarize the Stepping Stone method in following steps

1. Construct a transportation table with a given unit cost of transportation along with the rim conditions
2. Determine a initial basic feasible solution (allocation) using a suitable method as discussed earlier
3. Evaluate all unoccupied cells for the effect of transferring one unit from an occupied cell to the unoccupied cell. This transfer is made by forming a closed path that retains the SS and DD condition of the problem
4. Check the sign of each of the net change in the unit transportation costs. If the net changes are plus or zero, then the an optimal solution has been arrived at, otherwise go to step 5
5. Select the unoccupied cell with most negative net change among all unoccupied cells.
6. Assign many units as possible to unoccupied cell satisfying rim conditions. The maximum number of units to be assigned are equal to the smaller circled number among the occupied cells with the minus value in a closed path.
7. Go to step 3, and repeat the problem until all unoccupied cells are evaluated and the net change result in positive or zero.

Modified Distribution (MODI) method

MODI method is based on the concept of duality. We follow following steps in MODI method

1. For an initial basic feasible solution with (m+n-1) occupied cells, calculate ui and vj for row and columns To start with, any one of ui's and vj is assigned the value zero. Then complete the calculation of ui's and vj's for other rows and columns by using the relation cij = ui + vj for all occupied cells
2. For unoccupied cells, calculate opportunity cost dij by using the relationship dij = cij –(ui + vj) for all i and j
3. Examine sign of each dij i. If dij > 0 then current basic feasible solution is optimal ii. If dij = 0 then current basic feasible solution will remain unaffected but an alternative solution exist iii. If one or more dij < 0 the solution is not optimal. iv.
4. Reallocation of units by allocating units to the unoccupied cell, and calculate the new transportation cost.
5. Test the revised solution for optimality. The procedure terminates when all dij ≥ 0 for unoccupied cells.

Assignment Problem

The AP is a special type of LPP where assignees are being assigned to perform task. For example, the assignees might be employees who need to be given work assignments. However, the assignees might not be people. They could be machines or vehicles or plants or even time slots to be assigned tasks.

To fit the definition of an assignment problem, the problem need to formulate in a way that satisfies the following assumptions

1. The number of assignees and the number of tasks are the same
2. Each assignees is to be assigned to exactly one task
3. Each task is to performed by exactly one assignee
4. There is a cost cij associated with assignee i performing task j.
5. The objective is to determine how all n assignments should be made to minimise the total cost.

Any problem satisfying all these assumptions can be solved extremely efficiently by algorithm designed specifically for assignment problem.

Mathematical form

The mathematical model for the assignment problem uses the following decision variables

X_{ij} = 1 if assignee(worker) i perform task j

=0 if not.

Thus each x_{ij} is a binary variable (it has value 0 or 1).

Lets Z denots the total cost, the assignment problem model is

Min $Z= \sum_i \sum_j c_{ij} x_{ij}$

Subject to

$\sum_i x_{ij}=1$ for i= 1....n

$\sum_j x_{ij}=1$ for j=1.....n

$X_{ij} \geq 0$ for all i and j.

The first set of functional constraints specifies that each assignee is to perform exactly one task, whereas the second set requires each task to be performed by exactly one assignee.

Transportation Problem & Assignment Problem

In comparison to TP, in AP we have following restrictions

1. Number of sources (m) = number of destinations (n)

2. Each supply (Si) = 1

3. Each demand (di) = 1

Because assignment model is special case of transportation model so we can solve assignment problem directly as regular transportation model. Nevertheless, the fact that all the supply and demand amounts equal 1 has led to the development of a simple solution algorithm called the Hungarian Method.

The use of transportation Model technique in assignment model has following drawbacks

1. Wasted iterations- because in assignment model, the number of basic variable is n and all basic variables are binary variable so there always are n-1 degenerate basic variables. As we know that degenerate basic variable do not cause any major complication in the execution of the algorithm. However they do frequently cause wasted iterations.

2. Transportation simplex method is purely a general purpose algorithm for solving all transportation problems. Therefore, it does nothing to exploit the additional special structure in this special type of transportation problem (n=m, si=1 and di=1).

Hungarian Method

Hungarian Method is for assigning jobs by a one-for-one matching to identify the lowest-cost solution. Each job must be assigned to only one machine. It is assumed that every machine is capable of handling every job, and that the costs or values associated with each assignment combination are known and fixed. The number of rows and columns must be the same.

1. Subtract the smallest number in each row from every number in the row. This is called row reduction
2. Subtract the smallest number in each column of the new table from every number in the column. This is called column reduction.
3. Test whether an optimal assignment can be made. You do this by determining the minimum number of lines to cover all zeros. If the number of lines equals the number of rows, an optimal set of assignment is possible. Otherwise go on to step 4
4. If the number of lines is less than the number of rows, modify the table in the following way (a) Subtract the smallest uncovered number from every uncovered number in the table (b) Add the smallest uncovered number to the numbers at intersections of covering lines (c) Numbers crossed out but at the interactions of cross out lines carry over unchanged to the next table
5. Repeat step 3 and 4 until an optimal set of assignments is possible.

6. Make the assignments one at a time in positions that have zero elements. Begin with rows or columns that have only one zero. Since each row and each column needs to receive exactly one assignment, cross out both the row and the column involved after each assignment is made. Then move on to the rows and such row or column that are not yet crossed out to select the next assignment, with preference again given to any such row or column that has only one zero that is not crossed out. Continue until every row or column has exactly one assignment and so has been crossed out.